

Homework 10: on LRN Theorem and complex measure (40/40)

(Note: For this homework I applied for an one-day extension since I met with some emergent problem with my bank and rent payment.)

complex measure 的 total variation 的 formulas

Let ν be a complex measure on a measurable space (X, \mathcal{A}) . Prove that, for any $E \in \mathcal{A}$:

$$\begin{aligned} |\nu|(E) &= \sup\left\{\sum_{j=1}^n |\nu(E_j)| \mid n \in \mathbb{N}, E_1 \dots E_n \text{ disjoint}, E = \bigcup_{j=1}^n E_j\right\} \\ &= \sup\left\{\sum_{j=1}^{\infty} |\nu(E_j)| \mid E_1, E_2, \dots \text{ disjoint}, E = \bigcup_{j=1}^{\infty} E_j\right\} \\ &= \sup\left\{\left|\int_E f d\nu\right| \mid f: X \rightarrow \mathbb{C} \text{ measurable}, |f| \leq 1\right\}. \end{aligned}$$

Proof Take some positive measure μ s.t. $\nu \ll \mu$ (e.g. $\mu := |\operatorname{Re} \nu| + |\operatorname{Im} \nu|$), then by RN Thm there exists μ -unique RN derivative f , and $|\nu|$ can be defined by

$$d|\nu| := |f| d\mu$$

Now we denote:

$$\begin{aligned} \mu_1(E) &:= \sup\left\{\sum_{j=1}^n |\nu(E_j)| \mid n \in \mathbb{N}, E_1 \dots E_n \text{ disjoint}, E = \bigcup_{j=1}^n E_j\right\} \\ \mu_2(E) &:= \sup\left\{\sum_{j=1}^{\infty} |\nu(E_j)| \mid E_1, E_2, \dots \text{ disjoint}, E = \bigcup_{j=1}^{\infty} E_j\right\} \\ \mu_3(E) &:= \sup\left\{\left|\int_E f d\nu\right| \mid f: X \rightarrow \mathbb{C} \text{ measurable}, |f| \leq 1\right\}. \end{aligned}$$

We will prove the equality by showing that $\mu_1 \leq \mu_2 \leq |\nu|(E) \leq \mu_3 \leq \mu_1$.

Claim 1: $\mu_1 \leq \mu_2$.

Proof: This is trivial since for each finite disjoint segmentation $E = \bigsqcup_{j=1}^n E_j$ of E can be made into a countable segmentation of E , by taking all $E_N = \emptyset$ for $N \geq n + 1$. So every value included in $\{\sum_{j=1}^n |\nu(E_j)| \mid E = \bigsqcup_{j=1}^n E_j\}$ is also in $\{\sum_{j=1}^{\infty} |\nu(E_j)| \mid E = \bigsqcup_{j=1}^{\infty} E_j\}$. Thus taking sup, we have the ineq.

Claim 2: $\mu_2 \leq |\nu| \leq \mu_3$.

Since $\nu \ll |\nu|$ (Folland prop 3.13), by complex RN Thm we have have

$$f := \frac{d\nu}{d|\nu|} \in L^1(|\nu|)$$

Notice that f have absolute value 1, $|\nu|$ -a.e. (Folland prop 3.13)

Suppose $E = \sqcup_{j=1}^{\infty} E_j$, we have:

$$\begin{aligned}
\sum_{j=1}^{\infty} |\nu(E_j)| &\leq \sum_{j=1}^{\infty} |\nu|(E_j) && \text{by property of total variation measure} \\
&= |\nu|(E) = \int_E 1 d|\nu| && \text{by ctbl disjoint additivity} \\
&= \int_E |f|^2 d|\nu| = \int_E \bar{f} f d|\nu| && \text{since } f \text{ have absolute value } 1 \nu\text{-a.e.} \\
&= \int_E \bar{f} \frac{d\nu}{d|\nu|} d|\nu|
\end{aligned}$$

To confirm this equal to $\int \bar{f} d\nu$, we extend Folland prop 3.9 to the complex case.

Proposition 11.1

For complex measure ν and σ -finite positive measure μ s.t. $\nu \ll \mu$, if $g \in L^1(\nu)$, then

$$g\left(\frac{d\nu}{d\mu}\right) \in L^1(\mu), \quad \int g d\nu = \int g\left(\frac{d\nu}{d\mu}\right) d\mu$$

And the proof just follows from the finite signed-measure case, applied both to im part and re part.

$$\begin{aligned}
\int g d\nu &= \int g d(\operatorname{Re} \nu) + i \int g d(\operatorname{Im} \nu) \\
&= \int g\left(\frac{d(\operatorname{Re} \nu)}{d\mu}\right) d\mu + i \int g\left(\frac{d(\operatorname{Im} \nu)}{d\mu}\right) d\mu \\
&= \int g\left(\operatorname{Re} \frac{d\nu}{d\mu} + i \operatorname{Im} \frac{d\nu}{d\mu}\right) d\mu \\
&= \int g\left(\frac{d\nu}{d\mu}\right) d\mu
\end{aligned}$$

Now we back to Claim 2, since $f, \bar{f} \in L^1(\nu)$, we have:

$$\begin{aligned}
\sum_{j=1}^{\infty} |\nu(E_j)| &\leq |\nu|(E) \\
&= \int_E \bar{f} \frac{d\nu}{d|\nu|} d|\nu| \\
&= \int_E \bar{f} d\nu \\
&\leq \left| \int_E \bar{f} d\nu \right|
\end{aligned}$$

Since $|\bar{f}| \leq 1$ (in ν -a.e. sense), this shows that every element in $\{\sum_{j=1}^{\infty} |\nu(E_j)| \mid E = \sqcup_{j=1}^{\infty} E_j\}$ is less then or equal to $|\nu|(E)$, and $|\nu|(E)$ is less then some element in $\{|\int_E f d\nu| \mid \text{measurable } |f| \leq 1\}$, proves that $\mu_2 \leq |\nu| \leq \mu_3$.

Claim 3: $\mu_3 \leq \mu_1$.

For arbitrary simple function $\phi := \sum_{k=1}^n c_k \chi_{E_k}$ where $|c_k| \leq 1$ for all k , E_i s are disjoint and $\bigcup_{i=1}^n E_i = E$. We

have

$$\begin{aligned}
\left| \int_E \phi d\nu \right| &\leq \sum_{k=1}^n \left| c_k \int_{E_k} \chi_{E_k} d\nu \right| \\
&= \sum_{k=1}^n |c_k| |\nu(E_k)| \\
&\leq \sum_{k=1}^n |\nu(E_k)| \\
&\leq \mu_1(E)
\end{aligned}$$

Now we consider the general case: any measurable f .

Fix arbitrary measurable f s.t. $|f| \leq 1$, since it is measurable, we can choose seq of simple functions $(\phi_n)_1^\infty$ that approximate f pointwisely from below.

$$\lim_{n \rightarrow \infty} \phi_n = f$$

with

$$0 \leq |\phi_1| \leq |\phi_2| \leq \dots \leq |f|$$

Then $|f|$ as a dominating function for $(|\phi_n|)_n$, **by DCT** we obtain:

$$\int_E f d(\operatorname{Re} \nu) = \lim_{n \rightarrow \infty} \int_E \phi_n d(\operatorname{Re} \nu)$$

and

$$\int_E f d(\operatorname{Im} \nu) = \lim_{n \rightarrow \infty} \int_E \phi_n d(\operatorname{Im} \nu)$$

Thus

$$\begin{aligned}
\int_E f d\nu &= \int_E f d(\operatorname{Re} \nu) + i \int_E f d(\operatorname{Im} \nu) \\
&= \lim_{n \rightarrow \infty} \left(\int_E \phi_n d(\operatorname{Re} \nu) + i \int_E \phi_n d(\operatorname{Im} \nu) \right) \\
&= \lim_{n \rightarrow \infty} \int_E \phi_n d\nu
\end{aligned}$$

Since for each ϕ_n , we have $0 \leq |\phi_n(x)| \leq |f(x)| \leq 1$ for a.e. $x \in E$, we can apply the ineq we obtained that

$$\left| \int_E \phi_n d\nu \right| \leq \mu_1(E)$$

for each n . Thus taking limit we get:

$$\left| \int_E f d\nu \right| \leq \mu_1(E)$$

Taking supremum over f , proves that $\mu_3(E) \leq \mu_1(E)$.

Thus since we have shown $\mu_1 \leq \mu_2 \leq |\nu| \leq \mu_3 \leq \mu_1$, every inequality above is an equality, i.e.

$$\mu_1 = \mu_2 = \mu_3 = |\nu|$$

finishing the proof.

complex measure 与其 total variation measure 之间的关系: 整体即可决定局部

Let ν be a complex measure on a measurable space (X, \mathcal{A}) .

11.0.1 $\nu(X) = |\nu|(X) \iff \nu = |\nu| \iff \nu$ positive

- (i) $\nu(X) = |\nu|(X)$;
- (ii) ν is a (finite) positive measure;
- (iii) $\nu = |\nu|$.

Proof (ii) \implies (iii): If ν is positive then $\nu^- = 0$, so $\nu = |\nu| = \nu^+$.

(iii) \implies (i): Trivially true by taking $E = X$.

(i) \implies (ii): Take some positive measure μ s.t. $\nu \ll \mu$ (e.g. $\mu := |\operatorname{Re} \nu| + |\operatorname{Im} \nu|$), then by RN Thm there exists μ -unique RN derivative f , and $|\nu|$ can be defined by

$$d|\nu| := |f| d\mu$$

Then by def

$$\int f d\mu = \int |f| d\mu, \quad \text{i.e.} \quad \int \operatorname{Re} f d\mu + i \int \operatorname{Im} f d\mu = \int |f| d\mu$$

Since the right hand side is real, we have:

$$\int (|f| - \operatorname{Re} f) d\mu = 0$$

Note that, $|f| - \operatorname{Re} f$ is always nonnegative, so this implies that $\operatorname{Re} f = |f|$ μ -a.e.

Thus $\operatorname{Im} f = 0$ μ -a.e., so $f = |f|$ is real and positive μ -a.e. Thus

$$\nu(E) = \int_E f d\mu \in \mathbb{R}_+, \quad \forall E \in \mathcal{A}$$

finishing the proof that ν is a positive measure.

11.0.2 $|\nu(X)| = |\nu|(X) \iff \nu = \lambda|\nu|$ for some $|\lambda| = 1$

Prove that the following two conditions are equivalent:

- (i) $|\nu(X)| = |\nu|(X)$;
- (ii) there exists a complex number λ with $|\lambda| = 1$ such that $\nu = \lambda|\nu|$.

Proof (i) \implies (ii): Since $\nu \ll |\nu|$, by complex RN Thm we have RN derivative

$$h := \frac{d\nu}{d|\nu|} \in L^1(|\nu|)$$

Notice that h have absolute value 1, $|\nu|$ -a.e.

Then by def of RN derivative we have

$$\nu(X) = \int_X h d|\nu|$$

Thus

$$|\nu(X)| = \left| \int_X h d|\nu| \right| \leq \int_X |h| d|\nu| = \int_X 1 d|\nu| = |\nu|(X)$$

Since we have $|\nu(X)| = |\nu|(X)$, it implies that:

$$\left| \int_X h d|\nu| \right| = \int_X |h| d|\nu|$$

Claim: h is constant $|\nu|$ -a.e.

We first prove a lemma:

Lemma 11.1

Let μ be a finite positive measure.

For measurable function $f : X \rightarrow \mathbb{C}$, if $|f| = k$ a.e. for some nonzero constant k and

$$\left| \int f d\mu \right| = \int |f| d\mu$$

then f must be a.e. constant.



Proof of Lemma: Set:

$$c := \frac{\int f d\mu}{\left| \int f d\mu \right|}$$

Then $|c| = 1$, and we consider:

$$\int f d\mu = c \left| \int f d\mu \right| = c \int |f| d\mu$$

Define $g(x) := \bar{c}f(x)$, so:

$$\int g d\mu = \bar{c} \int f d\mu = \bar{c}c \int |f| d\mu = \int |f| d\mu$$

Notice $\int |f| d\mu \in \mathbb{R}_+$ and

$$\int g d\mu = \int \operatorname{Re} g d\mu + i \int \operatorname{Im} g d\mu \in \mathbb{C}$$

Thus

$$\int \operatorname{Re} g d\mu = \int |g| d\mu = \int |f| d\mu \implies \int (\operatorname{Re} g - |g|) d\mu = 0$$

Since by def:

$$0 \leq \operatorname{Re} g \leq |g|$$

We must have

$$\operatorname{Re} g = |g| \quad a.e.$$

This proves that g is a.e. real. And also since $|g| = |f| = k$ a.e., g is then constant k a.e.

Therefore, f is constant $\frac{k}{\bar{c}}$ a.e.

Now we go back to the proof of the original statement. By our Lemma we get:

$$h = \frac{\left| \int h d\nu \right|}{\int h d\nu} \quad \text{constant for } |\nu| \text{-a.e. } x$$

Therefore,

$$\nu = \frac{\left| \int h d\nu \right|}{\int h d\nu} |\nu|$$

This finishes the proof of (i) \implies (ii).

(ii) \implies (i): This direction is trivial. Since $\nu = \lambda|\nu|$, we have

$$|\nu(X)| = |\lambda||\nu|(X) = 1|\nu|(X) = |\nu|(X)$$

11.1 complex measures on (X, \mathcal{A}) 组成一个 complex Banach space

Let (X, \mathcal{A}) be a measurable space. Prove that the set \mathcal{M} of complex measures on (X, \mathcal{A}) is a complex Banach space, with norm given by $\|\nu\| := |\nu|(X)$.

Proof Claim 1: \mathcal{M} is a complex vector space, with addition operation defined by the addition of two complex measures, and scalar multiplication defined by scaling a complex measure by a complex number.

Proof of Claim 1: For $\nu, \mu \in \mathcal{M}$, and $\alpha \in \mathbb{C}$, define:

- $(\nu + \mu)(E) := \nu(E) + \mu(E)$ for all $E \in \mathcal{A}$
- $(\alpha\nu)(E) := \alpha \cdot \nu(E)$ for all $E \in \mathcal{A}$.

Then: $(\nu + \mu)(\emptyset) = 0 + 0 = 0$, $(\alpha\nu)(\emptyset) = \alpha 0 = 0$.

Also, $\nu + \mu$ and $\alpha\nu$ are both countably additive, since sum and scalar multiples preserve this property: for $E = \bigsqcup_{j=1}^{\infty} E_j$ with each $E_j \in \mathcal{A}$, we have:

$$(\nu + \mu)\left(\bigsqcup_{j=1}^{\infty} E_j\right) = \nu\left(\bigsqcup_{j=1}^{\infty} E_j\right) + \mu\left(\bigsqcup_{j=1}^{\infty} E_j\right) = \nu(E) + \mu(E) = (\nu + \mu)(E)$$

and

$$(\alpha\nu)\left(\bigsqcup_{j=1}^{\infty} E_j\right) = \alpha \cdot \nu\left(\bigsqcup_{j=1}^{\infty} E_j\right) = \alpha\nu(E)$$

So they are also complex measures, showing that \mathcal{M} is closed under addition and scalar multiplication, thus a complex vector space.

Claim 2: total variation $\|\nu\| := |\nu|(X)$ **defines a norm on** \mathcal{M} .

Proof of Claim 2: To verify this is a norm, we check the norm requirements:

- **Nonnegative:** $\|\nu\| \geq 0$, and $\|\nu\| = 0 \iff \nu = 0$

Proof: $\|\nu\| \geq 0$ follows from that $|\nu|$ is a p.m.

Since we know $\nu \ll |\nu|$, if $|\nu|(X) = 0$ then X is a null set of $|\nu|$, and thus is a null set for ν , so $\nu = 0$;

Conversely, if $\nu = 0$ then

$$\|\nu\| := |\nu|(X) = \sup\left\{\sum_{j=1}^n |\nu(E_j)| : X = \bigsqcup_{j=1}^n E_j\right\} = \sup\{0\} = 0$$

finishing the proof that $\|\nu\| = 0 \iff \nu = 0$

- **Homogeneity:** $\|\alpha\nu\| = |\alpha| \cdot \|\nu\|$

Proof:

$$\begin{aligned} \|\alpha\nu\| &:= |\alpha\nu|(X) = \sup\left\{\sum_{j=1}^n |\alpha\nu(E_j)| : X = \bigsqcup_{j=1}^n E_j\right\} \\ &= |\alpha| \sup\left\{\sum_{j=1}^n |\nu(E_j)| : X = \bigsqcup_{j=1}^n E_j\right\} \\ &= |\alpha| |\nu|(X) = |\alpha| \|\nu\| \end{aligned}$$

- **Triangle inequality:** $\|\nu + \mu\| \leq \|\nu\| + \|\mu\|$

Proof:

$$\begin{aligned}
 |\nu + \kappa|(X) &= \sup \left\{ \sum_{i=1}^n |(\nu + \kappa)(E_i)| : X = \bigsqcup_{i=1}^N E_i \right\} \\
 &\leq \sup \left\{ \sum_{i=1}^n (|\nu(E_i)| + |\kappa(E_i)|) : X = \bigsqcup_{i=1}^N E_i \right\} && \text{by tri ineq in } \mathbb{R} \\
 &= \sup \left\{ \sum_{i=1}^n |\nu(E_i)| + \sum_{i=1}^n |\kappa(E_i)| : X = \bigsqcup_{i=1}^N E_i \right\} \\
 &\leq \sup \left\{ \sum_{i=1}^n |\nu(E_i)| : X = \bigsqcup_{i=1}^N E_i \right\} + \sup \left\{ \sum_{i=1}^n |\kappa(E_i)| : X = \bigsqcup_{i=1}^N E_i \right\} \\
 &= |\nu|(X) + |\kappa|(X)
 \end{aligned}$$

Here we have finished the proof of $(\mathcal{M}, \|\cdot\|)$ being a normed \mathbb{C} -vector space.

Claim 3: $(\mathcal{M}, \|\cdot\|)$ is complete (thus Banach space)

Proof: Let (ν_n) be a Cauchy sequence in \mathcal{M} . We have

$$|\nu_n(B) - \nu_m(B)| = |(\nu_n - \nu_m)(B)| \leq |(\nu_n - \nu_m)(X)| = \|\nu_n - \nu_m\| \quad \text{for all } B \in \mathcal{A}$$

In particular, $(\nu_n(B))_n$ is a Cauchy sequence for all $B \in \mathcal{A}$. For each $B \in \mathcal{A}$, this is a Cauchy seq in \mathbb{C} , thus converges. So we can get:

$$\nu(B) := \lim_n \nu_n(B)$$

as the pointwise limit (by a point we mean a set).

Claim 3.1: $\nu \in \mathcal{M}$.

Since for all n , $\nu_n(\emptyset) = 0$, we have:

$$\nu(\emptyset) := \lim_n \nu_n(\emptyset) = 0$$

For a countable disjoint union of measurable sets $E = \bigsqcup_{i=1}^{\infty} E_i$,

$$\nu(E) = \lim_n \nu_n(E) = \lim_n \sum_i \nu_n(E_i)$$

We know by property of total variation measure that for each n we have:

$$\sum_i |\nu_n(E_i)| < |\nu_n|(X) = \|\nu_n\| < M$$

for some uniform bound M for each n , since $\|\nu_n\|$ is a Cauchy seq in \mathbb{C} . Thus we can exchange the order of taking limit and sum. Then we get:

$$\nu(E) = \lim_n \nu_n(E) = \lim_n \sum_i \nu_n(E_i) = \sum_i \lim_n \nu_n(E_i) = \sum_i \nu(E_i)$$

verifying the countable disjoint additivity.

And notice, as we have mentioned, for each measurable set $E \in \mathcal{A}$, since $(\nu_n(E))_n$ is a Cauchy sequence in \mathbb{C} , **it is bounded**, verifying that ν is a **valid complex measure**.

Claim 3.2: $\nu_n \rightarrow \nu$ in $\|\cdot\|$.

Fix $\epsilon > 0$.

By Cauchy in $\|\cdot\|$, there exists $N \in \mathbb{N}$ s.t. for all $m, n \geq N$, we have

$$\|\nu_m - \nu_n\| = |\nu_m - \nu_n|(X) < \epsilon$$

Fix $n \geq N$, and consider the sequence ν_m . Then $\nu_m \rightarrow \nu$ pointwise implies $\nu_n - \nu_m \rightarrow \nu_n - \nu$ **pointwise**. Thus

$$\|\nu_n - \nu\| = |\nu_n - \nu|(X) \leq \liminf_{m \rightarrow \infty} |\nu_n - \nu_m|(X) < \epsilon$$

Since $\epsilon > 0$ is arbitrary, this shows that, $\|\nu_n - \nu\| \rightarrow 0$ as $n \rightarrow \infty$, proving the convergence is in norm.

Now we conclude that $(\mathcal{M}, \|\cdot\|)$ is a Banach space.

Positivity

Let ν_1, ν_2 be complex measures on a measurable space (X, \mathcal{A}) such that $\|\nu_1 + \nu_2\| = \|\nu_1\| + \|\nu_2\|$. Is it true that there exists a nonzero constant $a \in \mathbb{C}$ such that $a\nu_1$ and $a\nu_2$ are both positive measures?

Sol. No, not necessarily.

Proof Consider $X := \{m, n\}$

Define ν_1, ν_2 by atoms:

$$\nu_1(\{m\}) = \nu_2(\{m\}) = 1, \quad \nu_1(\{n\}) = \nu_2(\{n\}) = -1$$

Then

$$\|\nu_1 + \nu_2\| = \|2\nu_1\| = |2\nu_1|(X) = 4 = \|\nu_1\| + \|\nu_2\|$$

But there is no nonzero constant $a \in \mathbb{C}$ such that $a\nu_1$ and $a\nu_2$ are both positive measures.

This is because for any nonzero constant a scaled on ν_1 : **if a real, then it either flip, or preserve the sign of $\nu_1(\{m\})$ and $\nu_1(\{n\})$, where there is always one positive number and one negative number between them; if a complex, then make the two numbers complex.**

In both case, ν_1 cannot become a positive measure. And since ν_2 is defined the same as ν_1 , same for it. Therefore it can never become positive measure by scaling a nonzero constant.

Averaging: Conditional Expectation

Let (X, \mathcal{A}, μ) be a finite measure space (i.e. a measure space such that $\mu(X) < \infty$). Let $\mathcal{B} \subset \mathcal{A}$ be a sub- σ -algebra, and set $\nu := \mu|_{\mathcal{B}}$. Thus (X, \mathcal{B}, ν) is also a finite measure space.

- Prove that if $f: X \rightarrow \mathbb{C}$ is \mathcal{B} -measurable, then f is \mathcal{A} -measurable. Is the converse true?
- Suppose that $f \in L^1(\mu)$. Prove that there exists a \mathcal{B} -measurable function $g \in L^1(\nu)$ such that $\int_E f d\mu = \int_E g d\nu$ for all $E \in \mathcal{B}$. Also prove that any two such functions g must agree outside a set of ν -measure zero.
- Construct g explicitly in the case when $X = \{1, 2, 3, 4\}$, $\mathcal{A} = \mathcal{P}(X)$, $\mu(\{i\}) = 1/4$ for $i \in X$, and $\mathcal{B} = \{\emptyset, \{1, 2\}, \{3, 4\}, X\}$. Thus, given the four complex numbers $f(i)$, $1 \leq i \leq 4$, you should find the four complex numbers $g(i)$, $1 \leq i \leq 4$.

Hint: use the Radon–Nikodym Theorem. *Remark:* if μ is a probability measure, then we can view g as the conditional expectation of (the random variable) f with respect to the σ -algebra \mathcal{B} .

Proof of (a): Suppose $f: X \rightarrow \mathbb{C}$ is \mathcal{B} -measurable, then for any Borel set $B \subset \mathbb{C}$, $f^{-1}(B) \in \mathcal{B} \subset \mathcal{A}$, so f is \mathcal{A} -measurable.

The converse is not true.

Consider $X = \{0, 1, 2, 3\}$, $\mathcal{A} := \mathcal{P}(X)$, $\mathcal{B} := \{\emptyset, X\}$.

Consider $f : x \mapsto x$ from X to \mathbb{R} .

f is \mathcal{A} -measurable since \mathcal{A} is the power set, containing all subsets of X .

But $f^{-1}(\{0\}) = \{0\} \notin \mathcal{B}$. Thus f is not \mathcal{B} -measurable.

Proof of (b): Let $\nu := \mu|_{\mathcal{B}}$, and define a signed measure on \mathcal{B} by:

$$\lambda(E) := \int_E f d\mu, \quad E \in \mathcal{B}$$

Then $\lambda \ll \nu$, since $\nu(E) = \mu(E) = 0 \implies \lambda(E) = 0$.

By Radon-Nikodym Thm, there exists a \mathcal{B} -measurable function $g \in L^1(\nu)$ such that

$$\lambda(E) = \int_E g d\nu \quad \text{for all } E \in \mathcal{B}$$

Then

$$\int_E f d\mu = \int_E g d\nu, \quad \forall E \in \mathcal{B}$$

Suppose g_1, g_2 are both such functions, then

$$\int_E (g_1 - g_2) d\nu = 0 \quad \forall E \in \mathcal{B}$$

Define

$$G^+ := \{g_1 - g_2 > 0\}, G^- := \{g_1 - g_2 < 0\}$$

These two sets are in \mathcal{B} since g_1, g_2 are \mathcal{B} -measurable. Then we have:

$$\int_{G^+} (g_1 - g_2) d\nu = \int_{G^-} (g_1 - g_2) d\nu = 0$$

Since on G^+ we have $g_1 - g_2 > 0$,

$$\int_{G^+} (g_1 - g_2) d\nu = 0 \implies \int_{G^+} |g_1 - g_2| d\nu = 0 \implies g_1 = g_2 \text{ } \nu\text{-a.e. on } G^+ \implies \nu(G^+) = 0$$

Similarly, since on G^- we have $g_1 - g_2 < 0$,

$$\int_{G^-} (g_1 - g_2) d\nu = 0 \implies -\int_{G^-} |g_1 - g_2| d\nu = 0 \implies g_1 = g_2 \text{ } \nu\text{-a.e. on } G^- \implies \nu(G^-) = 0$$

Thus

$$\nu\{g_1 \neq g_2\} = \nu(G^+) + \nu(G^-) = 0$$

This finishes the proof.

Sol. of (c): Given:

- $X = \{1, 2, 3, 4\}$
- $\mathcal{A} = \mathcal{P}(X)$
- $\mu(\{i\}) = 1/4$ for each i
- $\mathcal{B} = \{\emptyset, \{1, 2\}, \{3, 4\}, X\}$

Suppose we have: $f : X \rightarrow \mathbb{C}$, so $f(i) \in \mathbb{C}$ for $i = 1, 2, 3, 4$. We want to find: $g(i) \in \mathbb{C}$, $i = 1, 2, 3, 4$, such that g is \mathcal{B} -measurable and

$$\int_E f d\mu = \int_E g d\nu \quad \text{for all } E \in \mathcal{B}$$

Notice that $\mathcal{B} = \{\emptyset, \{1, 2\}, \{3, 4\}, X\}$, we must set $g(1) = g(2)$ and $g(3) = g(4)$, this is because, suppose if

we set $g(1) \neq g(2)$, then it will happen that

$$1 \in g^{-1}(g(1)) \not\cong 2$$

No set in \mathcal{B} satisfy this condition, thus $g^{-1}(g(1)) \notin \mathcal{B}$, contradicts that g is \mathcal{B} -measurable.

Thus we set

$$g(1) = g(2) = a, \quad g(3) = g(4) = b$$

We have:

$$\int_{\{1,2\}} g \, d\nu = \int_{\{1,2\}} f \, d\nu = f(1)\mu(\{1\}) + f(2)\mu(\{2\}) = \frac{f(1) + f(2)}{4}$$

and

$$\int_{\{3,4\}} g \, d\nu = \int_{\{3,4\}} f \, d\nu = f(3)\mu(\{3\}) + f(4)\mu(\{4\}) = \frac{f(3) + f(4)}{4}$$

while on the other hand

$$\int_{\{1,2\}} g \, d\nu = \frac{g(1) + g(2)}{4} = \frac{a}{2}, \quad \int_{\{3,4\}} g \, d\nu = \frac{g(3) + g(4)}{4} = \frac{b}{2}$$

Thus g is defined by:

$$g(1) = g(2) = \frac{f(1) + f(2)}{2}, \quad g(3) = g(4) = \frac{f(3) + f(4)}{2}$$

Thus what g expresses: is the conditional expectation of f on $\{1, 2\}, \{3, 4\}$.

(Therefore it can be generalized: given any sub σ -algebra $\mathcal{B} \subset \mathcal{A}$, there exists a $\mu|_{\mathcal{B}}$ -unique \mathcal{B} measurable function $g \in L^1(\mu|_{\mathcal{B}})$, that is the conditional expectation

$$g = \mathbb{E}[f \mid \mathcal{B}]$$

s.t. for $B \in \mathcal{B}$,

$$\int_B f \, d\mu = \int_B \mathbb{E}[f \mid \mathcal{B}] \, d\mu$$

it gives the average of f on sets in \mathcal{B} .)

Nur für Verrückte

(It's **really** not necessary to attempt these problems. Do not, under any circumstances, hand them in!) To any measure space (X, \mathcal{A}) we can associate a new measure space (Y, \mathcal{B}) , where Y is the Banach space of complex measures on (X, \mathcal{A}) , and \mathcal{B} is the Borel σ -algebra on Y .

- (a) Does this operation define a functor from the category of measurable spaces to itself. Is this functor (if well defined) full? Is it faithful? Is it essentially surjective?
- (b) Does the operation above admit any nontrivial fixed points (up to isomorphism)?